JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 24, No. 2, March–April 2001

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Conversion of North American Aerospace Defense Command Elements to Epicycle Elements

Yan Mai* and Philip Palmer†
University of Surrey,
Guildford, England GU2 7XH, United Kingdom

Introduction

THE epicycle description of an orbit[1] is an analytic formulation of the orbit of a satellite about an axisymmetric Earth. It has a simple analytic form, which is capable of describing all of the gravitational perturbative effects to $\mathcal{O}(10^{-7})$. Unlike more rigorous treatments, the epicycle approach has a simple geometric interpretation and greater mathematical simplicity than conventional descriptions of the perturbed motion, such as SGP4.[2] At the same time, it is also sufficiently accurate to describe the motion of a low-Earth-orbit (LEO) satellite over its lifetime for present-day applications.

To describe the orbit of a satellite requires knowledge of six parameters: the semimajor axis a; eccentricity e; inclination i; right ascension of the ascending node, Ω; argument of perigee, ω; and time of some perigee passage, t_p. Different descriptions of the orbit of a satellite rely on different definitions of these parameters, which often vary only very subtly. The most easily available data on the orbits of satellites is from NASA, who provides the North American Aerospace Defense Command (NORAD) elements. These are freely available by public Internet access‡ and provide accurate descriptions of satellite orbits. To use these elements, there is an orbit propagator SGP4, which is also freely available.

There are a number of advantages in having a simple analytic description of the orbits of satellites, mostly related to satellite autonomy. The autonomous determination of a satellite orbit is becoming viable with the success of satellite global positioning systems and an analytic formulation of the orbit greatly reduces the computational load of this task. Analytic forms for satellite orbits also enable the computation of orbit-related data such as the rise and set times over a particular location on the globe[3] that can be used by autonomous satellites for image capture or data download. Finally, autonomous orbit maneuvering as required in formation flying can be more efficiently achieved using good analytic models.

The purpose of this Note is to relate the NORAD orbital elements to the epicycle description of a satellite orbit so that these freely available data sets can be used either by SGP4 or in an epicycle formulation of satellite orbits. We first describe the relationship between the NORAD elements and the epicycle elements; we then compare results of propagating the orbit using both SGP4 and the epicycle approach to show the level of consistency achieved.

Received 8 May 2000; revision received 1 September 2000; accepted for publication 12 November 2000.

*Ph.D. Student, Surrey Space Centre.
†Senior Lecturer, Surrey Space Centre.
‡Data available online at http://www.celestrak.com [cited 27 January 2001].

Element Sets Conversion

NORAD‡ maintains general perturbation element sets for some resident space objects. These element sets are periodically refined to maintain a reasonable prediction capability on all space objects. In turn, these element sets are provided to users. The NORAD element sets are mean values obtained by removing periodic variations in a particular way. To obtain good predictions, these periodic variations must be reconstructed (by the prediction model) in exactly the same way they were removed by NORAD. Hence, inputting NORAD element sets into a different model (even though the model may be more accurate, such as a numerical integrator) will result in degraded predictions. The NORAD element set can only be used with one of the models described in Ref. 2. They are currently provided for the SGP4 users, for whom the value of the mean motion is slightly altered, and a pseudodrag term ($\dot{n}/2$, where $\dot{n}$ is the rate of change of mean motion) is generated. The element set consists of $\{n_N, e_N, i_N, \Omega_N, \omega_N, M_N\}$ which are the mean motion, eccentricity, inclination, right ascension of ascending node, argument of perigee, and mean anomaly at the epoch.

In the epicycle formulation, the satellite orbits around an axisymmetric Earth model, and therefore, energy is conserved. This is used to define the semimajor axis of the orbit. This is a constant along the orbit, which corresponds to the mean orbital radius used in SGP4. Hence, we can determine the semimajor axis from

$$\varepsilon = -(\mu/2a) \tag{1}$$

where ε is the specific orbital energy of the satellite and μ is the gravitational parameter. This is related to the NORAD mean motion through

$$a_E = \sqrt[3]{\mu/n_N} \tag{2}$$

where a_E is the epicycle semimajor axis.

The inclination and ascending node define an osculating orbital plane that contains the position and velocity vectors of the satellite, rather than a mean orbital plane as used in SGP4. We can relate the epicycle inclination and right ascension of the ascending node i_E, Ω_E to the NORAD parameters by incorporating the short periodic variations of the osculating orbital plane:

$$i_E = i_N - \tfrac{3}{8}J_2(R/a)^2 \sin 2i_E \cos 2\alpha_0 \tag{3}$$

$$\Omega_E = \Omega_N + \tfrac{3}{4}J_2(R/a)^2 \cos i_E \sin 2\alpha_0 \tag{4}$$

where R is the radius of the Earth and α_0 is related to the mean anomaly along the orbit. The epicycle phase α is measured from the ascending node of the satellite's orbit and so is approximately given by $\alpha = \omega_N + M_N$, where $\alpha = n_N t$.

There are two corrections that must be taken into account in determining α_0. The first is due to the eccentricity of the orbit, and the second is a small $\mathcal{O}(J_3)$ correction that gives a long periodic variation to the eccentricity. We incorporate these corrections into α_0 as follows:

$$\alpha_0 = \omega_N + M_N + (J_3/2J_2)(R/a)\sin i_E \cos\alpha_0 + e_N \sin(\alpha_0 - \alpha_P) \tag{5}$$

where α_P is the epicycle phase at the perigee passage. This is taken to be

$$\alpha_P = \omega_N \tag{6}$$

We need to solve Eqs. (3) and (5) simultaneously to find the epicycle parameters i_E and α_0. Once these have been solved for, we can find Ω_E.

The remaining parameter in the epicycle formulation is the eccentricity e_0. Because we have incorporated the long periodic effects into α_0 , then the eccentricity is just

$$e_0 = e_N \tag{7}$$

This completes the determination of the epicycle parameter set $\{a_E, e_0, i_E, \Omega_E, \alpha_P, \alpha_0\}$.

Results

To determine the accuracy of this conversion to the epicycle element set, we proceed as follows. Using the SGP4 propagator, we compute the satellite position and velocity at future times. In the epicycle formulation, the location of the satellite is given by the redundant set of coordinates (r, λ, i, Ω), where r is the radial distance of the satellite from the center of the Earth and λ is the argument of latitude measured on the orbital plane. We can access these quantities inside SGP4 because r, i, and Ω are all computed in the SGP4 iteration and λ is equated to the internal variable u. We compare the elements i and Ω with the predictions using the epicycle equations (Figs. 1 and 2).

Figures 1 and 2 show agreement of the orbital inclination to within 10^{-7} rad over elapsed times of 7 days. The right ascension of the ascending node has a secular drift due to differences in the computation of the secular drift. The magnitude of the difference in ascending node is less than 10^{-4} rad after 1 week, which corresponds to an error in the drift rate of 10^{-10} rad/s. This confirms a high level of agreement between SGP4 and the epicycle formulation for the orbital plane.

We next compared the position of the satellite on the orbital plane (r, λ) (Figs. 3 and 4).

Again we see a secular drift in the argument of latitude. This arises from the secular variation of the argument of perigee due to

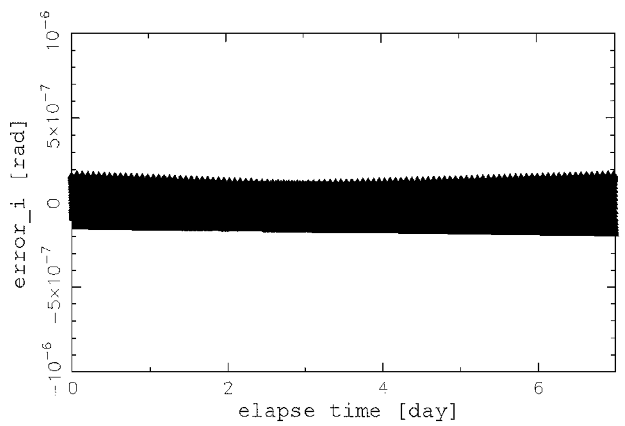

Fig. 1 Inclination comparison.

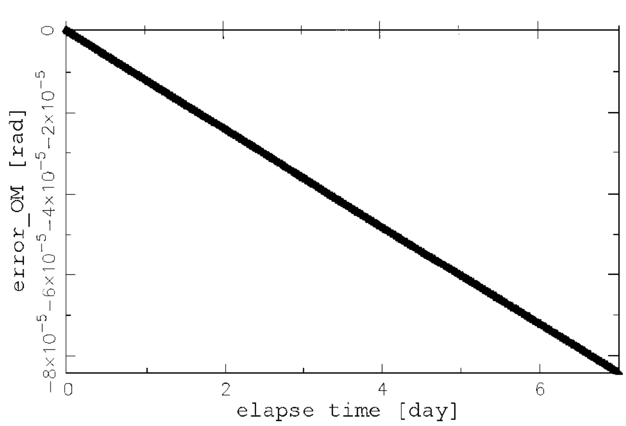

Fig. 2 Ascending node comparison.

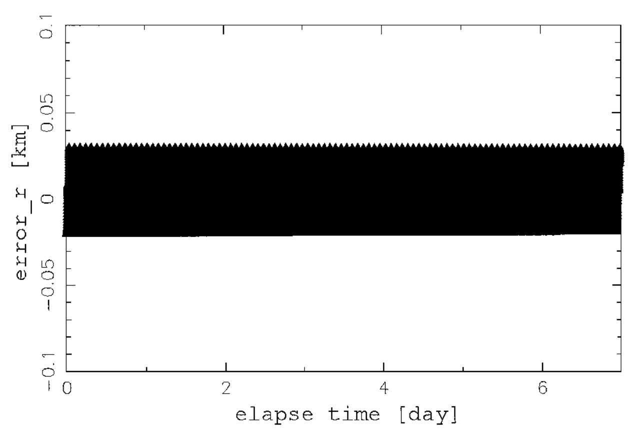

Fig. 3 Orbital radius comparison.

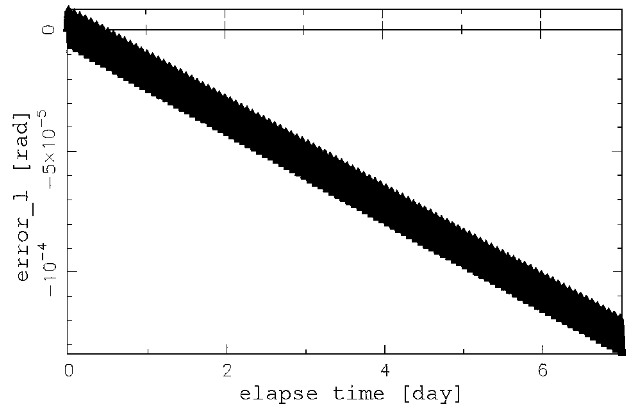

Fig. 4 Argument of latitude comparison.

the Earth's oblateness. The level of error in argument of latitude after 7 days is roughly 10^{-4} rad, which shows a similar level of error in the secular term to that found in the ascending node.

We note that the argument of latitude has a constant offset error when compared with SGP4 of order 10^{-4} rad. Unlike the secular term discussed earlier, this error does not grow in time, and so represents an acceptable error. We have determined that this error arises from the determination of the mean anomaly in SGP4. The argument of perigee is determined with the inclusion of J_4 and J_2^2 terms, but the mean anomaly includes only the J_2^2 term. Looking back at the original derivation of these formulas in Brouwer's paper[4] shows a J_4 term should have been included. The effect of this missing term, however, is negligibly small.

Finally, we compare the orbital radius between SGP4 and the epicycle equations (Fig. 3). From Fig. 3, we see that the difference in orbital radius between the two models is on the order of $1 \sim 30$ m and that this discrepancy exists right at the initial epoch. A satellite in LEO has a radial distance from Earth of about 7000 km, and so a $1 \sim 30$ m error corresponds to 10^{-6} error in precision. We have investigated the SGP4 model and find that when SGP4 calculates the orbital radius it includes only a J_2^2 term but no J_4 term. We believe the exclusion of the J_4 term causes this 10^{-6} error.

We have tried to verify the semimajor axis of the orbit by computing the orbital energy in SGP4 and deriving the semimajor axis from Eq. (1). We find, however, that the energy is not conserved in SGP4 and that the variation in the energy is of order 10^{-6}. This small energy error arises from the missing J_4 terms. Because SGP4 is a single precision propagator, this level of error is consistent with the machine accuracy.

Conclusions

We have investigated the SGP4 model and compared it with the epicycle formulation for satellite orbits. We have shown that SGP4 is formulated only to single precision accuracy and cannot be extended to higher accuracy without the addition of more complexity in the formulas used. We have shown how to convert the NORAD element set into epicycle parameters used in the analytic epicycle formulation of an orbit. This formulation is accurate to 10^{-7} and can be extended to higher levels of accuracy if required. We have shown that the correspondence between the NORAD elements and the epicycle parameters provides high levels of accuracy in satellite prediction over a timescale of 7 days.

As a result of this work, it is now possible to use a more accurate analytic model of satellite orbits and have access to orbital parameters through the widespread availability of NORAD data sets. The analytic models can be used on the ground for mission analysis tasks, Internet access to satellite data, and onboard satellites for autonomous operation.

References

[1]Hashida, Y., and Palmer, P., "Epicyclic Motion of Satellites about an Oblate Planet," *Journal of Guidance, Control, and Dynamics* (to be published).

[2]Hoots, F. R., and Roehrich, R. L., "Models for Propagation of NORAD Element Sets," SPACETRACK Rept. 3., Aerospace Defence Center, Peterson AFB, Colorado Springs, CO, 1980, pp. 10–20.

[3]Escobal, P., *Methods of Orbit Determination*, Wiley, New York, 1976, pp. 162–174.

[4]Brouwer, D., "Solution of the Problem of Artificial Satellite Theory Without Drag," *Astronomical Journal*, Vol. 64 , No. 1274, 1959, pp. 378–397.

Closed-Form Solution of Line-of-Sight Trajectory for Maneuvering Target

Rachad M. Shoucri*
Royal Military College of Canada,
Kingston, Ontario K7K 7B4, Canada

I. Introduction

IN previous studies[1–4] the differential equation

$$\ddot{\boldsymbol{r}} + f\dot{\boldsymbol{r}} + g\boldsymbol{r} = 0 \tag{1}$$

was used to describe planar elliptical trajectories under a noncentral force field, where f and g are scalar functions of the polar variables r and θ and/or their derivatives. It was shown that for planar motion

$$f = -(\dot{l}/l) \tag{2}$$

where $l = r^2\dot{\theta}$ is the angular momentum and dots indicate differentiation with respect to time. The application of Eq. (1) is extended here to the study of the trajectories of a homing missile under various guidance laws. It is shown that scattered published results in the literature can all be derived from Eq. (1). Moreover, a trajectory equation is derived for a homing missile in the case where the target velocity and the missile velocity are varying.

II. Guidance Law

Equation (1) can be easily reduced to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\boldsymbol{r}}}{l}\right) = -\frac{g}{l}\boldsymbol{r} \tag{3}$$

where $\boldsymbol{r}$ is the vector corresponding to the line-of-sight distance $\boldsymbol{r} = \boldsymbol{r}_m - \boldsymbol{r}_t$, $\boldsymbol{r}_m$ is the missile distance, and $\boldsymbol{r}_t$ is the target distance

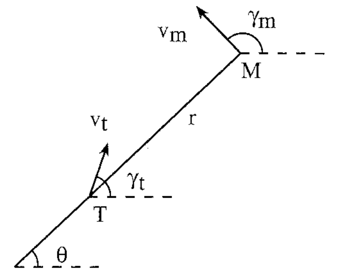

Fig. 1 Scenario for a missile-target interception.

from the origin (Fig. 1). Equation (3) can be written in components form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_r}{l}\right) - \frac{\dot{\theta}}{l}v_\theta = -\frac{g}{l}r \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v_\theta}{l}\right) + \frac{\dot{\theta}}{l}v_r = 0 \tag{5}$$

where $v_r = \dot{r}$ is the radial component of the velocity and $v_\theta = r\dot{\theta}$ is the transverse component of the velocity. In the case where l is constant, Eqs. (4) and (5) reduce to the case corresponding to the closed-form solution of generalized proportional navigation treated by Yang et al.[5] If the pursuer is always on the line of sight, then a collision will result if the polar angles of the missile and target are such that $\theta_m(t) = \theta_t(t) = \theta(t)$. Equation (3) can also be written in a form giving the radial and transverse components of the acceleration

$$\ddot{r} - r\dot{\theta}^2 = -gr + (\dot{l}/l)\dot{r} \tag{6}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = (\dot{l}/l)r\dot{\theta} \tag{7}$$

By differentiating $x = r\cos\theta$ and $y = r\sin\theta$, one obtains the x component $\dot{x} = v\cos\gamma$ and the y component $\dot{y} = v\cos\gamma$ of the velocity vector $\dot{\boldsymbol{r}}$. They can be written as

$$v\cos\gamma = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \tag{8}$$

$$v\sin\gamma = \dot{r}\cos\theta + r\dot{\theta}\cos\theta \tag{9}$$

with

$$\tan\gamma = \frac{\dot{r}\sin\theta + r\dot{\theta}\cos\theta}{\dot{r}\cos\theta - r\dot{\theta}\sin\theta} \tag{10}$$

where γ is the angle between the direction of the velocity v and the x axis. By differentiating Eq. (10) with respect to θ, one obtains the following equation:

$$\frac{1}{\cos^2\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\theta} = \frac{r^2 + 2(\mathrm{d}r/\mathrm{d}\theta)^2 - r(\mathrm{d}^2r/\mathrm{d}\theta^2)}{[(\mathrm{d}r/\mathrm{d}\theta)\cos\theta - r\sin\theta]^2} \tag{11}$$

By noting that

$$\frac{1}{\cos^2\gamma} = \frac{(\mathrm{d}r/\mathrm{d}\theta)^2 + r^2}{[(\mathrm{d}r/\mathrm{d}\theta)\cos\theta - r\sin\theta]^2} \tag{12}$$

we finally get

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\theta} = 2 - \frac{r^2 + r(\mathrm{d}^2r/\mathrm{d}\theta^2)}{r^2 + (\mathrm{d}r/\mathrm{d}\theta)^2} \tag{13}$$

When Eqs. (6) and (7) are used, Eq. (13) yields

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\theta} = \frac{g}{\dot{\theta}^2}\sin^2\psi \tag{14}$$

with

$$\sin^2\psi = r^2 \bigg/ \left[r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\right] \tag{15}$$

where $\psi = \gamma - \theta$ is the angle between v and r. Equation (14) gives a guidance law between γ and θ (or ψ and θ) corresponding to the

Received 14 August 2000; revision received 24 September 2000; accepted for publication 26 September 2000.

*Professor, Department of Mathematics and Computer Science; shoucri-r@rmc.ca.